CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough, J.-Y. Enjalbert, P. Simonnet.

CIP seminar, Friday conversations:,

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory.
4. MRS factorisation: A local system of coordinates for Hausdorff groups and fine tuning between analysis and algebra.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimers.

Disclaimer.– The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

Disclaimer II.– The reader will find repetitions and reprises from the preceding CCRT[n], they correspond to some points which were skipped or uncompletely treated during preceding seminars.

CCRT[25] Kleene stars and shuffle algebras II.

A (tangled) tale of various (bi-)algebras

The goal of this talk (number II) is threefold:

1. A first shot about linear independence of characters of enveloping algebras w.r.t. some algebras of nilpotents (Mathoverflow), extends to bialgebras (cocommutative or not), two proofs. This result is one of the three variations of a general theme [4].
2. Application to algebraic independence of some group of series w.r.t. polynomials (built on formal power series).
3. More on the structure Hausdorff groups: One-parameter groups, local system of coordinates, identities, motivations ...

Conclusion(s): More applications and perspectives.

Outline

Initial motivation (one of)

Lappo-Danilevskij's setting

J. A. Lappo-Danilevskij (J. A. Lappo-Danilevsky), Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. I, *Travaux Inst. Physico-Math. Stekloff*, 1934, Volume 6, 1–256

§ 2. HYPERLOGARITHMES 159

§ 2. Hyperlogarithmes. En abordant la résolution algorithmique du problème de Poincaré, nous introduisons le système des fonctions

$$L_b(a_{j_1}, a_{j_2}, \ldots, a_{j_\nu} | x), \qquad (j_1, j_2, \ldots, j_\nu = 1, 2, \ldots, m;\ \nu = 1, 2, 3 \ldots)$$

définies par les relations de récurrence:

$$
(10) \quad
\begin{aligned}
L_b(a_{j_1} | x) &= \int_b^x \frac{dx}{x - a_{j_1}} = \log \frac{x - a_{j_1}}{b - a_{j_1}}; \\
L_b(a_{j_1} a_{j_2} \ldots a_{j_\nu} | x) &= \int_b^x \frac{L_b(a_{j_1} \ldots a_{j_{\nu-1}} | x)}{x - a_{j_\nu}} dx,
\end{aligned}
$$

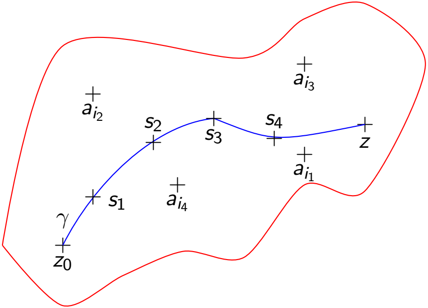
où b est un point fixe à distance finie, distinct des points $a_1, a_2, \ldots, a_m$. Ces fonctions seront nommées *hyperlogarithmes de la première espèce de*

Initial motivation (one of)/2

Let $(a_i)_{1\le i\le n}$ be a family of complex numbers (all different) and $z_0 \notin \{a_i\}_{1\le i\le n}$, then

Definition [Lappo-Danilevskij, 1928]

$$L(a_{i_1},\ldots,a_{i_n}|z_0 \overset{\gamma}{\rightsquigarrow} z) = \int_{z_0}^{z}\int_{z_0}^{s_n}\ldots\Big[\int_{z_0}^{s_1}\frac{ds}{s-a_{i_1}}\Big]\ldots\frac{ds_n}{s_n-a_{i_n}}.$$



Remarks

1. The result depends only on the homotopy class of the path and then the result is a holomorphic function on $\widetilde{B}$ $(B = \mathbb{C} \setminus \{a_1, \cdots, a_n\})$
2. From the fact that these functions are holomorphic, we can also study them in an open (simply connected) subset (a section) like the following cleft plane Ω

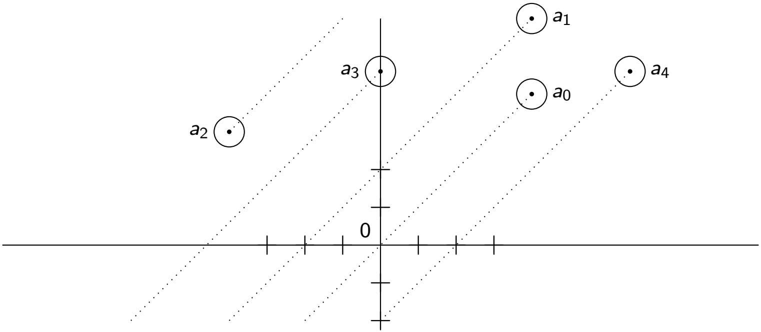


Figure: The cleft plane.

Remarks/2

3. The set of functions

$$\alpha_{z_0}^{z}(\underbrace{x_{i_1} \ldots x_{i_n}}_{word}) = L(\underbrace{a_{i_1}, \ldots, a_{i_n}}_{list} | z_0 \overset{\gamma}{\rightsquigarrow} z)$$

(or $1_{\mathcal{H}(B)}$ if the word is void) has a lot of nice combinatorial properties through its generating series

$$\sum_{w \in X} \alpha_{z_0}^{z}(w)\, w$$

- Noncommutative DE with left multiplier $\rightarrow$ Shuffle morphism.
- Linear independence $\rightarrow$ to be extended to larger sets of scalars.
- Factorisation $\rightarrow$ as characters.
- Possiblity of left or right multiplicative renormalization. at a neighbourhood of the singularities.
- Extension to (some) series.

Domain of $HL_\bullet$ (hyperlogarithms).

We now have an arrow of commutative algebras

$$(\mathbb{C}\langle X\rangle, \text{ш}, 1_{X^*}) \xrightarrow{HL_\bullet} (\mathcal{H}(\Omega), \times, 1_\Omega)$$

on the left $\mathbb{C}\langle X\rangle \hookrightarrow \mathbb{C}\langle\langle X\rangle\rangle$ is endowed with the Krull topology (coefficientwise stationary convergence) and, on the right $\mathcal{H}(\Omega)$ is endowed with the (Fréchet) topology of compact convergence.
We are led to the following definition.

Definition [Domain of $HL_\bullet$]

We define $Dom(HL_\bullet;\Omega)$ (or $Dom(HL_\bullet)$ if the context is clear) as the set of series $S = \sum_{n\geq 0} S_n$ (where $S_n = \sum_{|w|=n}\langle S|w\rangle\, w$, i.e. the decomposition is done by homogeneous slices) such that $\sum_{n\geq 0} HL_\bullet(S_n, z)$ converges unconditionally[a] for the compact convergence in Ω. One then sets $HL_\bullet(S, z) := \sum_{n\geq 0} HL_\bullet(S_n, z)$.

[a] In order to use functional properties of $\mathcal{H}(\Omega)$.

Domain of $HL_\bullet/2$

Diagram

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xhookrightarrow{HL_\bullet} & \mathbb{C}\{HL(w,z)\}_{w\in X^*} (= Span_{\mathbb{C}}\{HL(w,z)\}_{w\in X^*}) \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}\langle\langle X\rangle\rangle \supset Dom(HL_\bullet) & \longrightarrow & \mathcal{H}(\Omega)
\end{array}$$

Proposition

With this definition, we have

1. $Dom(HL_\bullet)$ is a shuffle unital subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(HL_\bullet) := Dom(HL_\bullet) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(HL_\bullet)$, we have

$$HL_\bullet(S ш T) = HL_\bullet(S).HL_\bullet(T) \text{ and } HL_\bullet(1_{X^*}) = 1_{\mathcal{H}(\Omega)}$$

Particular case: The ladder of polylogarithms

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Domain of $\mathrm{Li}_\bullet$ (particular case of $Dom(HL_\bullet)$)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S = \sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges for the compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \tag{1}$$

Examples

$$Li_{x_0^*}(z) = z,\ Li_{x_1^*}(z) = (1-z)^{-1}\ ;\ Li_{(\alpha x_0+\beta x_1)^*}(z) = z^{\alpha}(1-z)^{-\beta}$$

Useful properties

Star of the plane property

Every conc-character is of the form $\left(\sum_{x\in X} \alpha(x)\, x\right)^*$

We will see that, with the common pattern (3 first examples)

$$w \,\text{ш}_\varphi 1_{X^*} = 1_{X^*} \,\text{ш}_\varphi w = w \text{ and}$$

$$au \,\text{ш}_\varphi bv = a(u \,\text{ш}_\varphi bv) + b(au \,\text{ш}_\varphi v) + \varphi(a,b)(u \,\text{ш}_\varphi v)$$

We get the following examples

Shuffle: $(\alpha x)^* \text{ш} (\beta y)^* = (\alpha x + \beta y)^*$ $(\varphi \equiv 0)$

Stuffle: $(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$ $(\varphi(y_i, y_j) = y_{i+j})$

q-**infiltration**:
$(\alpha x)^* \uparrow_q (\beta y)^* = (\alpha x + \beta y + \alpha\beta q\delta_{x,y} x)^*$ $(\varphi(x,y) = q\delta_{x,y} x)$

Hadamard: $(\alpha a)^* \odot (\beta b)^* = 1_{X^*}$ if $a \neq b$ and $(\alpha a)^* \odot (\beta a)^* = (\alpha\beta a)^*$

Name	Formula (recursion)	φ	Reference
Shuffle	$au ш bv = a(u ш bv) + b(au ш v)$	$\varphi \equiv 0$	Ree
Stuffle	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_i u \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$	Hoffman
Min-stuffle	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) - x_{i+j}(u \sqcup v)$	$\varphi(x_i, x_j) = -x_{i+j}$	Costermans
Muffle	$x_i u \odot x_j v = x_i(u \odot x_j v) + x_j(x_i u \odot v) + x_{i \times j}(u \odot v)$	$\varphi(x_i, x_j) = x_{i \times j}$	Enjalbert,HNM
q-shuffle	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	Bui
q-shuffle$_2$	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q^{i.j} x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	Bui
$\mathtt{LDIAG}(1, q_s)$	$au ш bv = a(u ш bv) + b(au ш v) + q_s^{\lvert a\rvert\lvert b\rvert} a.b(u ш v)$	$\varphi(a, b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$	GD,Koshevoy,Penson,Tollu
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$	Chen-Fox-Lyndon
AC-stuffle	$au ш_\varphi bv = a(u ш_\varphi bv) + b(au ш_\varphi v) + \varphi(a, b)(u ш_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	Enjalbert,HNM
Semigroup--stuffle	$x_t u ш_\perp x_s v = x_t(u ш_\perp x_s v) + x_s(x_t u ш_\perp v) + x_{t \perp s}(u ш_\perp v)$	$\varphi(x_t, x_s) = x_{t \perp s}$	Deneufchâtel
φ-shuffle	$au ш_\varphi bv = a(u ш_\varphi bv) + b(au ш_\varphi v) + \varphi(a, b)(u ш_\varphi v)$	$\varphi(a, b)$ law of AA	Manchon, Paycha

Common pattern

$$w ш_\varphi 1_{X^*} = 1_{X^*} ш_\varphi w = w \text{ and}$$

$$au ш_\varphi bv = a(u ш_\varphi bv) + b(au ш_\varphi v) + \varphi(a, b)(u ш_\varphi v)$$

Independence of characters w.r.t. polynomials.

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Independence of characters with respect to polynomials

Ask Question

Asked 2 years, 2 months ago   Active 5 months ago   Viewed 305 times

6

2

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U}\to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+=\ker(\epsilon)$. We build the following filtrations ($N\geq 0$)

$$\mathcal{U}_N=\mathcal{U}_+^N=\underbrace{\mathcal{U}_+\ldots\mathcal{U}_+}_{N\text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0=\mathcal{U},\mathcal{U}_{N+1}=\mathcal{U}.\mathcal{U}_N$) and, for $N\geq -1$

Enveloping algebras in context.

1. Let $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ be two categories and $F : \mathcal{C}_{right} \to \mathcal{C}_{left}$ a (covariant) functor between them

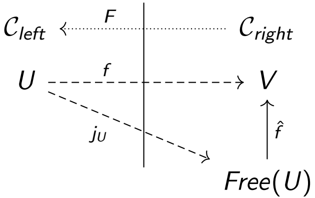


Figure: A solution of the universal problem w.r.t. the functor F is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$).

$$\big(\forall f \in Hom(U, F[V])\big)\big(\exists!\ \hat{f} \in Hom(Free(U), V)\big)\big(F(\hat{f}) \circ j_U = f\big)$$

2. In the case of enveloping algebras $\mathcal{C}_{left} = \mathbf{k} - \mathbf{Lie}$, $\mathcal{C}_{right} = \mathbf{k} - \mathbf{AAU}$ and $F(\mathcal{A})$ is the algebra $\mathcal{A}$ endowed with the bracket $[x, y] = xy - yx$ thus a Lie algebra.

Convolution of endomorphisms

1. If C is a $\mathbf{k}$-coalgebra, and if A is a $\mathbf{k}$-algebra, then the $\mathbf{k}$-module $\operatorname{Hom}(C, A)$ itself becomes a $\mathbf{k}$-algebra using a multiplication operation known as convolution. We denote it by $\circledast$, and recall how it is defined: For any two $\mathbf{k}$-linear maps $f, g \in \operatorname{Hom}(C, A)$, we have

$$f \circledast g = \mu_A \circ (f \otimes g) \circ \Delta_C : C \to A.$$

   The map $\eta_A \circ \epsilon_C : C \to A$ is a neutral element for this operation $\circledast$.

2. Let $\varphi \in \operatorname{Hom}_{bialg}(\mathcal{B}_1, \mathcal{B}_2)$ and f_i, g_i such that $g_i \varphi = \varphi f_i$ i.e.

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{\ \varphi\ } & \mathcal{B}_2 \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_i} \\ \mathcal{B}_1 & \xrightarrow{\ \varphi\ } & \mathcal{B}_2 \end{array}$$

3. then $(g_1 \circledast g_2)\varphi = \varphi(f_1 \circledast f_2)$

4. We have the following theorem
Theorem.– Let $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \varepsilon)$ be a bialgebra, then
   A) $\mathcal{B} = \ker(\epsilon) \oplus \mathbf{k}.1_{\mathcal{B}}$ and the projectors are
   i) $h \mapsto I_+(h) = h - \epsilon(h).1_{\mathcal{B}}$ on $\ker(\epsilon) = \mathcal{B}_+$.
   ii) $h \mapsto \mathrm{e}(h) = \epsilon(h).1_{\mathcal{B}}$ on $\mathbf{k}.1_{\mathcal{B}}$.
   B) If I_+ is locally nilpotent i.e.

$$(\forall b \in \mathcal{B})(\exists N \geq 0)(\forall n \geq N)(I_+^{*\,n}(b) = 0) \tag{2}$$

   then $\mathcal{B}$ is a Hopf algebra.
   C) (CQMM) If $\mathbb{Q} \subset \mathbf{k}$ and Δ is cocommutative, then TFAE
   i) $\mathcal{B}$ is an enveloping bialgebra.
   ii) $\mathcal{B} = \mathcal{U}(Prim(\mathcal{B}))$.
   iii) $\Delta_+ = I_+^{\otimes\, 2} \circ \Delta$ is locally nilpotent.
   iv) I_+ is locally nilpotent.

Sketch of proofs

5. (A) is easy to prove by direct computation.
6. (B) is the beginning of the HFC[a] because $I = Id = \mathrm{e} + I_+$. if I_+ is locally nilpotent, then Id is $\circledast$-invertible, ideed, for every $b \in \mathcal{B}$

$$(\mathrm{e} + I_+)^{\circledast\, -1} = \mathrm{e} - I_+ + (I_+)^{\circledast\, 2} - (I_+)^{\circledast\, 3} + (I_+)^{\circledast\, 4} - \cdots \tag{3}$$

7. (C) use HFC with

$$\mathcal{B}_1 = \mathcal{B};\ \mathcal{B}_2 = \mathcal{B} \otimes \mathcal{B};\ \varphi = \Delta_{\mathcal{B}}$$

then use points 2 and 3 with $\log_{\circledast}$ to prove that its image is within $Prim(\mathcal{B})$ and remark that $\log_{\circledast}$ is the identity when restricted to $Prim(\mathcal{B})$. Conclude remarking that, then, $\mathcal{B}$ is primitively generated.

[a] Holomorphic Functional Calculus.

Independence of characters w.r.t. polynomials./2

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra. We note ϵ its counit and set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations ($N \geq 0$)

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N \geq -1$

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^\perp = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing (in particular $\mathcal{U}_{-1}^* = \{0\}$, $\mathcal{U}_0^* = k.\epsilon$).
One shows easily that, for $p, q \geq 0$ (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 0} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Independence of characters w.r.t. polynomials./3

Now, we can state the

Theorem (From MO, k ring without zero divisors)

The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}_{\infty}^{*}$.

Remark

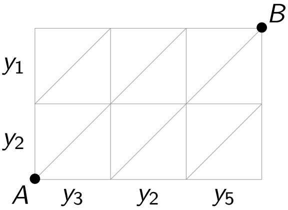
i) $\mathcal{U}_{\infty}^{*}$ is a commutative k-algebra.
ii) The title ("Independence of characters ...") comes from the fact that, with $(k\langle X\rangle, conc, 1)$ (non commutative polynomials), k a $\mathbb{Q}$-algebra (without zero divisors) and one of the usual comultiplications (with Δ_{+} cocommutative and nilpotent, as co-shufflle, co-stuffle or - commutatively - deformed), if one takes $\mathfrak{g}$ as the space of primitive elements, we have $\mathcal{U}^{*} = k\langle\langle X\rangle\rangle$ (series) and $\mathcal{U}_{\infty}^{*} = k\langle X\rangle$.

Examples as found in the literature.

Name	Formula (recursion)	φ	Reference
Shuffle	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$	Ree
Stuffle	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_i u \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$	Hoffman
Min-stuffle	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) - x_{i+j}(u \sqcup v)$	$\varphi(x_i, x_j) = -x_{i+j}$	Costermans
Muffle	$x_i u \odot x_j v = x_i(u \odot x_j v) + x_j(x_i u \odot v) + x_{i \times j}(u \odot v)$	$\varphi(x_i, x_j) = x_{i \times j}$	Enjalbert,HNM
q-shuffle	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	Bui
q-shuffle$_2$	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q^{i.j} x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	Bui
$\mathtt{LDIAG}(1, q_s)$	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v) + q_s^{\|a\|\|b\|} a.b(u \shuffle v)$	$\varphi(a, b) = q_s^{\|a\|\|b\|}(a.b)$	GD,Koshevoy,Penson,Tollu
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$	Chen-Fox-Lyndon
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	Enjalbert,HNM
Semigroup- -stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v) + x_{t \perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t \perp s}$	Deneufchâtel
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b)$ law of AAU	Manchon, Paycha

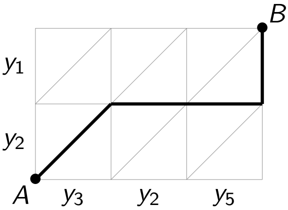
$$w \shuffle_\varphi 1_{X^*} = 1_{X^*} \shuffle_\varphi w = w \text{ and}$$
$$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a, b)(u \shuffle_\varphi v)$$

With $Y = \{y_i\}_{i \geq 1}$, one can see the product $u ш_\varphi v$ as a sum indexed by paths (with right-up-diagonal steps) within the grid formed by the two words (u horizontal and v vertical, the diagonal steps corresponding to the factors $\varphi(a, b)$)

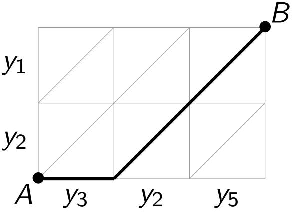


Computation of $y_2 y_1 ш_\varphi y_3 y_2 y_5$

For example, the path



evaluates as $\varphi(y_2, y_3) y_2 y_5 y_1$

the path  reads $y_3\varphi(y_2,y_2)\varphi(y_1,y_5)$.

We have the following

Theorem (Radford theorem for $ш_\varphi$)

Let $\mathbf{k}$ *be a* $\mathbb{Q}$*-algebra (associative, commutative with unit) such that*

$$ш_\varphi : \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle$$

is associative and commutative then

- $(\mathcal{L}yn(X)^{ш_\varphi \alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ *is a linear basis of* $\mathbf{k}\langle X\rangle$.
- *This entails that* $(\mathbf{k}\langle X\rangle, ш_\varphi, 1_{X^*})$ *is a polynomial algebra with* $\mathcal{L}yn(X)$ *as transcendence basis.*

Making (combinatorial) bialgebras

Proposition

Let $\mathbf{k}$ *be a commutative ring (with unit). We suppose that the product* φ *is associative, then, on the algebra* $(\mathbf{k}\langle X\rangle, \text{ш}_{\varphi}, 1_{X^*})$*, we consider the comultiplication* Δ_{conc} *dual to the concatenation*

$$\Delta_{conc}(w) = \sum_{uv=w} u \otimes v \tag{4}$$

and the "constant term" character $\varepsilon(P) = \langle P|1_{X^*}\rangle$*.*
Then

(i) *With this setting, we have a bialgebra* [a]*.*

$$\mathcal{B}_{\varphi} = (\mathbf{k}\langle X\rangle, \text{ш}_{\varphi}, 1_{X^*}, \Delta_{conc}, \varepsilon) \tag{5}$$

(ii) *The bialgebra (eq. 5) is, in fact, a Hopf Algebra.*

[a] Commutative and, when $|X| \geq 2$, noncocommutative.

Dualizability

If one considers φ as defined by its structure constants

$$\varphi(x,y) = \sum_{z\in X} \gamma_{x,y}^{z}\, z$$

one sees at once that ш_φ is dualizable within $\mathbf{k}\langle X\rangle$ iff the tensor $\gamma_{x,y}^{z}$ is locally finite in its contravariant place "z" i.e.

$$(\forall z \in X)(\#\{(x,y)\in X^2 | \gamma_{x,y}^{z} \neq 0\} < +\infty)\ .$$

Remark

Shuffle, stuffle and infiltration are dualizable. The comultiplication associated with the stuffle with negative indices is not.

Dualizability/2

In the case when $ш_\varphi$ is dualizable, one has a comultiplication

$$\Delta_{ш_\varphi} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$$

such that, for all $u, v, w \in X^*$

$$\langle u \, ш_\varphi v | w\rangle = \langle u \otimes v | \Delta_{ш_\varphi}(w)\rangle \tag{6}$$

Then, the following

$$\mathcal{B}_\varphi^\vee = (\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{ш_\varphi}, \varepsilon) \tag{7}$$

is a bialgebra in duality with $\mathcal{B}_\varphi$ (not always a Hopf algebra although $\mathcal{B}$ was so, for example, see $\mathcal{B}$ with $ш_\varphi = \uparrow_q$ i.e. the q-infiltration).

The interest of these bialgebras is that they provide a host of easy-to-within-compute bialgebras with easy-to-implement-and-compute set of characters. Some of them are enveloping algebras.

CQMM: examples and counterexamples.

8. Let $\mathbf{k}$ be a ring, S be a subsemigroup of $\mathbb{N}$ and, for $s \in S$, $\Delta_{\uplus}(y_s) := \sum_{p+q=s} y_p \otimes y_q$, then

$$\mathcal{B} = \mathcal{B}_{\uplus} = (\mathbf{k}\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus}, \epsilon) \tag{8}$$

is a bialgebra.

   i) If $S = \mathbb{N}_{\geq 1}$ (classical stuffle) and $\mathbf{k} = \mathbb{Z}$ $\mathcal{B}_{\uplus}$ is not an enveloping algebra.
   ii) With $S = \mathbb{N}$ and even $\mathbf{k} = \mathbb{Q}$ (we called this alphabet Y_0 in the Ph. D's), $\mathcal{B}_{\uplus}$ is not even a Hopf algebra.

9. **Remarks.–**

   i) (Weak form of the CQMM) With $\mathbb{Q} \subset \mathbf{k}$ and $\mathcal{B}$, connected, graded and cocommutative.
   Rq.– This, strictly weaker, form doesn't cover classical enveloping algebras as $\mathcal{U}(sl_2(\mathbf{k}))$.
   ii) In the equivalent conditions of CQMM, $\log(I) = \log(e + I_+)$ is the π_1 projector $\mathcal{B} \to Prim(\mathcal{B})$.

Proposition (Conc-Bialgebras)

Let $\mathbf{k}$ *be a commutative ring,* X *a set and* $\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^{z}\,z$ *an associative and dualizable law on* $\mathbf{k}.X$*. Let* $ш_{\varphi}$ *and* $\Delta_{ш_{\varphi}}$ *be the associated product and co-product. Then:*
i) $\mathcal{B}=(\mathbf{k}\langle X\rangle,conc,1_{X^*},\Delta_{ш_{\varphi}},\epsilon)$ *is a bialgebra which, in case* $\mathbb{Q}\hookrightarrow\mathbf{k}$*, is an enveloping algebra iff* φ *is commutative and* $\Delta^{+}_{ш_{\varphi}}$ *nilpotent.*
ii) In the general case $S\in\mathbf{k}\langle\langle X\rangle\rangle=\mathbf{k}\langle X\rangle^{\vee}$ *is a character for* $\mathcal{A}=(\mathbf{k}\langle X\rangle,conc,1_{X^*})$ *(i.e. a conc-character) iff it is of the form*

$$S=\Big(\sum_{x\in X}\alpha_x\,x\Big)^*=\sum_{n\geq 0}\Big(\sum_{x\in X}\alpha_x\,x\Big)^n \text{ and, with this notation} \tag{9}$$

$$\Big(\sum_{x\in X}\alpha_x\,x\Big)^*ш_{\varphi}\Big(\sum_{x\in X}\beta_y\,y\Big)^*=\left(\sum_{z\in X}(\alpha_z+\beta_z)\,z+\sum_{x,y\in X}\alpha_x\beta_y\,\varphi(x,y)\right)^* \tag{10}$$

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Main result about independence of characters w.r.t.

Theorem (G.D., Darij Grinberg, H. N. Minh)

Let $\mathcal{B}$ be a $\mathbf{k}$*-bialgebra. As usual, let $\Delta = \Delta_{\mathcal{B}}$ and $\epsilon = \epsilon_{\mathcal{B}}$ be its comultiplication and its counit.*
Let $\mathcal{B}_+ = \ker(\epsilon)$. For each $N \geq 0$, let $\mathcal{B}_+^N = \underbrace{\mathcal{B}_+ \cdot \mathcal{B}_+ \cdot \cdots \cdot \mathcal{B}_+}_{N \text{ times}}$, where $\mathcal{B}_+^0 = \mathcal{B}$. Note that $\left(\mathcal{B}_+^0, \mathcal{B}_+^1, \mathcal{B}_+^2, \ldots\right)$ is called the standard decreasing filtration of $\mathcal{B}$.
For each $N \geq -1$, we define a $\mathbf{k}$*-submodule $\mathcal{B}_N^\vee$ of $\mathcal{B}^\vee$ by*

$$\mathcal{B}_N^\vee = (\mathcal{B}_+^{N+1})^\perp = \left\{ f \in \mathcal{B}^\vee \mid f\left(\mathcal{B}_+^{N+1}\right) = 0 \right\}. \qquad (11)$$

Thus, $\left(\mathcal{B}_{-1}^\vee, \mathcal{B}_0^\vee, \mathcal{B}_1^\vee, \ldots\right)$ is an increasing filtration of $\mathcal{B}_\infty^\vee := \bigcup_{N \geq -1} \mathcal{B}_N^\vee$ with $\mathcal{B}_{-1}^\vee = 0$.

Theorem (DGM, cont'd)

Let also $\Xi(\mathcal{B})$ be the monoid (group, if $\mathcal{B}$ is a Hopf algebra) of characters of the algebra $(\mathcal{B}, \mu_{\mathcal{B}}, 1_{\mathcal{B}})$.
Then:

(a) *We have $\mathcal{B}_p^\vee \circledast \mathcal{B}_q^\vee \subseteq \mathcal{B}_{p+q}^\vee$ for any $p, q \geq -1$ (where we set $\mathcal{B}_{-2}^\vee = 0$). Hence, $\mathcal{B}_\infty^\vee$ is a subalgebra of the convolution algebra $\mathcal{B}^\vee$.*

(b) *Assume that $\mathbf{k}$ is an integral domain. Then, the set $\Xi(\mathcal{B})^\times$ of invertible characters (i.e., of invertible elements of the monoid $\Xi(\mathcal{B})$) is left $\mathcal{B}_\infty^\vee$-linearly independent.*

Remark

The standard decreasing filtration of $\mathcal{B}$ is weakly decreasing, it can be stationary after the first step. An example can be obtained by taking the universal enveloping bialgebra of any simple Lie algebra (or, more generally, of any perfect Lie algebra); it will satisfy $\bigcap_{n\geq 0} \mathcal{B}_+^n = \mathcal{B}_+$.

Corollary

We suppose that $\mathcal{B}$ is cocommutative, and $\mathbf{k}$ *is an integral domain. Let* $(g_x)_{x \in X}$ *be a family of elements of* $\Xi(\mathcal{B})^\times$ *(the set of invertible characters of* $\mathcal{B}$*), and let* $\varphi_X : \mathbf{k}[X] \to (\mathcal{B}^\vee, \circledast, \epsilon)$ *be the* $\mathbf{k}$*-algebra morphism that sends each* $x \in X$ *to* g_x*. In order for the family* $(g_x)_{x \in X}$ *(of elements of the commutative ring* $(\mathcal{B}^\vee, \circledast, \epsilon)$*) to be algebraically independent over the subring* $(\mathcal{B}^\vee_\infty, \circledast, \epsilon)$*, it is necessary and sufficient that the monomial map*

$$\begin{aligned} m : \mathbb{N}^{(X)} &\to (\mathcal{B}^\vee, \circledast, \epsilon), \\ \alpha &\mapsto \varphi_X(X^\alpha) = \prod_{x \in X} g_x^{\alpha_x} \end{aligned} \tag{12}$$

(where α_x *means the* x*-th entry of* α*) be injective.*

Examples

Let **k** be an integral domain, and let us consider the standard bialgebra $\mathcal{B} = (\mathbf{k}[x], \Delta, \epsilon)$ For every $c \in \mathbf{k}$, there exists only one character of $\mathbf{k}[x]$ sending x to c; we will denote this character by $(c.x)^* \in \mathbf{k}[[x]]$ (motivation about this notation is Kleene star). Thus, $\Xi(\mathcal{B}) = \{(c.x)^* \mid c \in \mathbf{k}\}$. It is easy to check that $(c_1.x)^* \sqcup\!\sqcup (c_1.x)^* = \big((c_1 + c_2).x\big)^*$ for any $c_i \in \mathbf{k}$ (‡). Thus, any $c_1, c_2, \ldots, c_k \in \mathbf{k}$ and any $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ satisfy

$$\begin{aligned} &\left((c_1.x)^*\right)^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup \left((c_2.x)^*\right)^{\sqcup\!\sqcup\, \alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left((c_k.x)^*\right)^{\sqcup\!\sqcup\, \alpha_k} \\ &= \big((\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k).x\big)^* . \end{aligned} \tag{13}$$

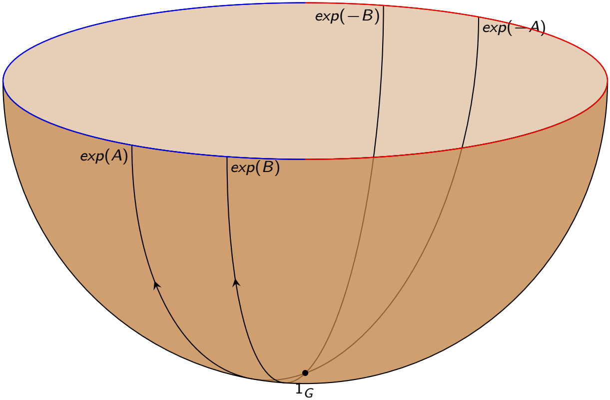
From (‡) above, the monoid $\Xi(\mathcal{B})$ is isomorphic with the abelian group $(\mathbf{k}, +, 0)$; in particular, it is a group, so that $\Xi(\mathcal{B})^\times = \Xi(\mathcal{B})$.

Examples/2

Take $\mathbf{k} = \overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$) and $c_n = \sqrt{p_n} \in \mathbf{k}$, where p_n is the n-th prime number. What precedes shows that the family of series $\big((\sqrt{p_n}x)^*\big)_{n\geq 1}$ is algebraically independent over the polynomials (i.e., over $\overline{\mathbb{Q}}[x]$) within the commutative $\overline{\mathbb{Q}}$-algebra $\big(\overline{\mathbb{Q}}[[x]], ш, 1\big)$. This example can be double-checked using partial fractions decompositions as, in fact, $(\sqrt{p_n}x)^* = \dfrac{1}{1-\sqrt{p_n}x}$ (this time, the inverse is taken within the ordinary product in $\mathbf{k}[[x]]$) and

$$\Big(\frac{1}{1-\sqrt{p_n}x}\Big)^{ш\, n} = \frac{1}{1-n\sqrt{p_n}x}.$$

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{\text{ш}}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

Hausdorff group of the stuffle Hopf algebra.

With $Y = \{y_i\}_{i\geq 1}$ and

$$\Delta_{\text{⧢}}(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j$$

the bialgebra $\mathcal{B} = (\mathbf{k}\langle X\rangle, \text{conc}, 1_{X^*}, \Delta_{\text{⧢}}, \epsilon)$ is an enveloping algebra (it is cocommutative, connex and graded by the weight function given by $||y_{i_1}y_{i_2}\cdots y_{i_k}|| = \sum_{s=1}^{k} i_s$ on a word $w = y_{i_1}y_{i_2}\cdots y_{i_k}$).
With $\varphi(y_i, y_j) = y_{i+j}$, (eq.10) gives

$$(\sum_{i\geq 1}\alpha_i\, y_i)^*_{\text{⧢}}(\sum_{j\geq 1}\beta_j\, y_j)^* = (\sum_{i\geq 1}\alpha_i\, y_i + \sum_{j\geq 1}\beta_j\, y_j + \sum_{i,j\geq 1}\alpha_i\beta_j\, y_{i+j})^* \tag{14}$$

This formula suggests us to code, in an umbral style, $\sum_{k\geq 1}\alpha_k\, y_k$ by the series $\sum_{k\geq 1}\alpha_k\, x^k \in \mathbf{k}_+[[x]]$. Indeed, we get the following proposition whose first part, characteristic-freely describes the group of characters $\Xi(\mathcal{B})$ and its law and the second part, about the exp-log correspondence, requires $\mathbf{k}$ to be $\mathbb{Q}$-algebra.

Proposition

Let π_Y^{Umbra} *be the linear isomorphism* $\mathbf{k}_+[[x]] \to \widehat{\mathbf{k}.Y}$ *defined by*

$$\sum_{n\geq 1} \alpha_n\, x^n \mapsto \sum_{k\geq 1} \alpha_k\, y_k \tag{15}$$

Then

1. *One has, for* $S, T \in \mathbf{k}_+[[x]]$,

$$(\pi_Y^{Umbra}(S))^* ⧢ (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \tag{16}$$

2. *From now on* $\mathbf{k}$ *is supposed to be a* $\mathbb{Q}$*-algebra.*
   For $t \in \mathbf{k}$ *and* $T \in \mathbf{k}_+[[x]]$, *the family* $(\frac{(t.T)^n}{n!})_{n\geq 0}$ *is summable and one sets*

$$G(t) = (\pi_Y^{Umbra}(e^{t.T}-1))^* \tag{17}$$

Proposition (Cont'd)

3. *The parametric character G fulfills the "stuffle one-parameter group" property i.e. for $t_1, t_2 \in \mathbf{k}$, we have*

$$G(t_1 + t_2) = G(t_1) \uplus G(t_2) \; ; \; G(0) = 1_{Y^*} \tag{18}$$

4. *We have*

$$G(t) = exp_{\uplus}\left(t.\pi_Y^{Umbra}(T)\right) \tag{19}$$

5. *In particular, calling π_x^{Umbra} the inverse of π_Y^{Umbra} we get, for $P^* \in \Xi(\mathcal{B})$ (in other words $P \in \widehat{\mathbf{k}.Y}$),*

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1 + \pi_x^{Umbra}(P))) \tag{20}$$

Proof (Sketch)

i) We have

$$\pi_Y^{Umbra}(S) = \sum_{i\geq 1}\langle S|x^i\rangle y_i \quad \pi_Y^{Umbra}(T) = \sum_{j\geq 1}\langle T|x^j\rangle y_j$$

and then

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\sum_{i\geq 1}\langle S|x^i\rangle y_i)^* \uplus (\sum_{j\geq 1}\langle T|x^j\rangle y_j) =$$
$$\Big(\sum_{i\geq 1}\langle S|x^i\rangle y_i) + \sum_{j\geq 1}\langle T|x^j\rangle y_j + \sum_{i,j\geq 1}\langle S|x^i\rangle\langle T|x^j\rangle y_{i+j}\Big)^* =$$
$$(\pi_Y^{Umbra}(S+T+ST))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^*$$

ii.1) The one parameter group property is a consequence of (16) applied to the series $e^{t_i.T} - 1,\ i = 1, 2$.

Proof (Sketch)/2

ii.2) Property 18 holds for every $\mathbb{Q}$-algebra, in particular in $\mathbf{k}_1 = \mathbf{k}[t]$ and $\mathbf{k}_1\langle\langle Y\rangle\rangle$ is endowed with the structure of a differential ring by term-by-term derivations (see [2] for formal details). We can write $G(t) = 1 + t.G_1 + t^2.G_2(t)$ (where $G_1 = \pi_Y^{Umbra}(T)$ is independent from t) and from 18, we have

$$G'(t) = G_1.G(t)\ ;\ \ G(0) = 1_{Y^*} \tag{21}$$

but $H(t) = exp_{\uplus}(t.G_1)$ satisfies 21 whence the equality.
ii.3) At $t = 1$, we have $\exp_{\uplus}(\pi_Y^{Umbra}(T)) = (\pi_Y^{Umbra}(e^T - 1))^*$ hence, with $P = \pi_Y^{Umbra}(e^T - 1)$ (take $T := \log(\pi_x^{Umbra}(P) + 1)$)

$$\pi_Y^{Umbra}(T) = \log_{\uplus}(P^*) \qquad \text{[QED]} \tag{22}$$

Application of (20)

$$(ty_k)^* = \exp_{\uplus}\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}t^n y_{nk}}{n}\Big) \tag{23}$$

Conclusion(s): More applications and perspectives.

1. Star of the plane property (slide 13) holds for non-commutative valued (as matrix-valued) characters.
2. Combinatorial study of other $ш_\varphi$ one-parameter groups and evolution equations in convolution algebras.
3. Factorisation of $\mathcal{A}$-valued characters ($\mathcal{A}$ **k**-CAAU).
   For example, with

$$\mathcal{B} = (\mathbf{k}\langle X\rangle, ш\,, 1_{X^*}, \Delta_{conc}, \epsilon)\,,\ \mathcal{A} = (\mathbf{k}\langle X\rangle, ш\,, 1_{X^*})\,,\ \chi = Id$$

(χ is a shuffle character) one has (MRS factorisation)

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w)\otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \qquad (24)$$

MRS : (Mélançon, Reutenauer, Schützenberger)

Conclusion(s): More applications and perspectives./2

4. Deformed version of factorisation above for ш_φ (with φ associative, commutative, dualisable and moderate). With

$$\mathcal{B} = (\mathbf{k}\langle X\rangle, \text{ш}_\varphi, 1_{X^*}, \Delta_{conc}, \epsilon)\,,\ \mathcal{A} = (\mathbf{k}\langle X\rangle, \text{ш}_\varphi, 1_{X^*})\,,\ \chi = Id$$

(χ is a shuffle character) one has

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w)\otimes w = \sum_{w\in X^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \qquad (25)$$

5. Holds for all enveloping algebras which are free as **k**-modules (with $\mathbb{Q}\to\mathbf{k}$). This could help to the combinatorial study of the group of characters of enveloping algebras of Lie algebras like KZ[a]-Lie algebras and other ones, or deformed.

[a] Knizhnik–Zamolodchikov.

THANK YOU FOR YOUR ATTENTION

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